A bound for the order of the fundamental group of a complete noncompact Ricci shrinker

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For shrinking (gradient) Ricci solitons, we note a quantification of Wylie's [6] result that $\pi_1(\mathcal{M}) < \infty$.

Lemma 1 (f-uniqueness) *Let $(\mathcal{M}^n, g, f_i, \varepsilon)$, $i = 1, 2$, be a complete gradient Ricci soliton (GRS), i.e., $\mathrm{Rc} + \nabla^2 f_i + \frac{\varepsilon}{2} g = 0$. Then $f_1 - f_2$ is constant or $(\mathcal{M}, g)$ is isometric to $(\mathbb{R}, ds^2) \times (\mathcal{N}^{n-1}, h)$, where $(\mathcal{N}, h)$ is isometric to each level set $\{f_1 - f_2 = c\}$, $c \in \mathbb{R}$. Moreover, $(f_1 - f_2)(s, y) = as + b$, $a, b \in \mathbb{R}$.*

Proof. Let $F = f_1 - f_2$. Then $\nabla^2 F = 0$. Assume that F is not a constant. Then $|\nabla F| = a$, where $a \in \mathbb{R}^+$. Let $\Sigma_c = \{F = c\}$. Then Σ_c is C^∞ and its second fundamental form is $\mathrm{II}(X, Y) = \frac{\nabla^2 F(X,Y)}{|\nabla F|} = 0$. Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be the 1-parameter group of isometries generated by ∇F. We have $F \circ \varphi_t = F + a^2 t$. Moreover, φ_t maps Σ_c isometrically onto $\Sigma_{c+a^2 t}$. This produces the product structure on $(\mathcal{M}, g)$. ■

Let $(\mathcal{M}^n, g, f)$ be a complete noncompact shrinking GRS, where $\mathrm{Rc} + \nabla^2 f = \frac{1}{2} g$ and f is normalized by $R + |\nabla f|^2 = f$. Let $(\tilde{\mathcal{M}}^n, \tilde{g}, \tilde{f})$ be the universal covering shrinking GRS, that is, $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ is the universal covering map, $\tilde{g} = \pi^* g$, and $\tilde{f} = f \circ \pi$. Then we have that $\mathrm{Rc}_{\tilde{g}} + \nabla^2_{\tilde{g}} \tilde{f} = \frac{1}{2} \tilde{g}$ and $R_{\tilde{g}} + |\nabla \tilde{f}|^2_{\tilde{g}} = \tilde{f}$.

Let $\gamma \in \pi_1(\mathcal{M}) - \{e\}$. Then γ corresponds to a deck transformation $\gamma : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ that is a fixed-point-free isometry of the metric $\tilde{g}$. Since both $\tilde{f}$ and $\tilde{f} \circ \gamma$ are normalized potential functions for $\tilde{g} = \gamma^* \tilde{g}$, by the lemma we have $\tilde{f} \circ \gamma = \tilde{f}$. In particular, γ is a fixed-point-free isometry of any level or sublevel set of $\tilde{f}$. Hence $\mathrm{Vol}_{\tilde{g}}(\{\tilde{f} \leq s\}) = |\pi_1(\mathcal{M})| \, \mathrm{Vol}_g(\{f \leq s\})$ for any $s > 0$. Let O be a minimum point of f. By Cao and Zhou [1] and its improvement of constants by Haslhofer and Müller [2], we have

$$\frac{1}{4} \left((d(x, O) - 5n)_+ \right)^2 \leq f(x) \leq \frac{1}{4} (d(x, O) + \sqrt{2n})^2 \tag{1}$$

for g and the same inequalities for $\tilde{g}$ (using $\tilde{f}$ and using a lift $\tilde{O}$ of O), and $\mathrm{Vol}_{\tilde{g}}(\{\tilde{f} \leq s\}) \leq C(n) s^{n/2}$. Similarly, Munteanu and Wang [4] proved that $\mathrm{Vol}\, B_r(O) \leq \omega_n e^{n/2} r^n$ for $r > 0$, where $\omega_n = \mathrm{Vol}_{\mathbb{R}^n} B_1$. By Munteanu and Wang [3], there is a constant $c(n, \int_{\mathcal{M}} e^{-f} d\mu) > 0$ such that $\mathrm{Vol}\, B_r(O) \geq cr$ for $r \geq 1$.

Proposition 2 *If $(\mathcal{M}^n, g, f)$ is a complete noncompact shrinking GRS, then $|\pi_1(\mathcal{M})| \leq C(n, \int_{\mathcal{M}} e^{-f} d\mu)$.*

Proof. By the above, we have that $C(n)(2n)^{\frac{n}{2}} \geq \mathrm{Vol}_{\tilde{g}}(\{\tilde{f} \leq 2n\}) = |\pi_1(\mathcal{M})| \, \mathrm{Vol}_g(\{f \leq 2n\})$. The proposition follows from this and the inequality $\mathrm{Vol}_g(\{f \leq 2n\}) \geq \mathrm{Vol}_g(B_{\sqrt{2n}}(O)) \geq c(n, \int_{\mathcal{M}} e^{-f} d\mu)$. ■

Remark 3 *Assume for $x \in \mathcal{M}$ and $r > 0$ such that $R \leq r^{-2}$ in $B_r(x)$, we have $\mathrm{Vol}\, B_s(x) \geq \kappa s^n$ for $0 < s \leq r$. (For shrinking GRS that are singularity models, there is such a $\kappa > 0$ by Perelman [5].) Since $R \leq 2n$ on $B_{\frac{1}{\sqrt{2n}}}(O)$, we have $\mathrm{Vol}\, B_{\frac{1}{\sqrt{2n}}}(O) \geq (2n)^{-\frac{n}{2}} \kappa$ and we conclude that $|\pi_1(\mathcal{M})| \leq C(n) \kappa^{-1}$.*

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